

Formation of compression waves with multiscale asymptotics in the Burgers and KdV models

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Abstract. The Cauchy problem for the Burgers equation with a small dissipation and an initial weak discontinuity and the Cauchy problem with a large initial gradient for a quasilinear parabolic equation and for the Korteweg–de Vries (KdV) equation are considered. Multiscale asymptotics of solutions corresponding to shock waves are constructed. Some results can also be applied to rarefaction waves.

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1 Wave with an initial weak discontinuity

A simplest model of the motion of continuum, which takes into account nonlinear effects and dissipation, is the equation of nonlinear diffusion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad \varepsilon > 0, \quad (1.1)$$

for the first time presented by J. Burgers [1]. This equation is used in studying the evolution of a wide class of physical systems and probabilistic processes, for example, acoustic waves in fluid and gas [2, 3]. The problem of shock wave formation (for equation (1.1) with $\varepsilon > 0$) is investigated in [3, 4, 5]. A perturbed weak discontinuity is considered in [6]. The problem of shock wave propagation for the Hopf type equation ($\varepsilon = 0$) is studied in [7, 8].

Below, we briefly state results of paper [4] for the following initial data:

$$u(x, -1, \varepsilon) = -(x + ax^2)\Theta(-x), \quad x \in \mathbb{R}^1, \quad (1.2)$$

where $a > 0$, $\Theta(x)|_{x \geq 0} = 1$, $\Theta(x)|_{x < 0} = 0$. The solution of the limit ($\varepsilon = 0$) equation is found using the method of characteristics:

$$u_0(x, t) = \begin{cases} \frac{2a(1+t)x + t + \sqrt{t^2 - 4a(1+t)x}}{2a(1+t)^2}, & x < s(t)\Theta(t), \\ 0, & x > s(t)\Theta(t). \end{cases}$$

The function u_0 has a jump discontinuity on the line $x = s(t)$, where

$$s(t) \equiv \frac{(3t+4)^{3/2} - 9t - 8}{9a(1+t)}, \quad s(t) = \frac{3t^2}{16a} - \frac{27t^3}{128a} + O(t^4), \quad t \rightarrow 0. \quad (1.3)$$

The solution of problem (1.1)–(1.2) is given by the following expression:

$$u(x, t, \varepsilon) = -2\varepsilon[\Psi(x, t, \varepsilon)]^{-1}\Psi_x(x, t, \varepsilon), \quad (1.4)$$

where

$$\begin{aligned} \Psi(x, t, \varepsilon) = & \frac{1}{2\sqrt{\pi(1+t)}} \int_{-\infty}^0 \exp \left\{ \frac{1}{\varepsilon} \left[-\frac{(x-y)^2}{4(1+t)} + \frac{y^2}{4} + \frac{ay^3}{6} \right] \right\} dy + \\ & + \frac{1}{2\sqrt{\pi(1+t)}} \int_0^{\infty} \exp \left\{ \frac{1}{\varepsilon} \left[-\frac{(x-y)^2}{4(1+t)} \right] \right\} dy. \end{aligned}$$

Taking into account these formulas, we write expression (1.4) in the form

$$u(x, t, \varepsilon) = -\frac{\Psi^0(x, t, \varepsilon)}{\Psi^-(x, t, \varepsilon) + \Psi^+(x, t, \varepsilon)}, \quad (1.5)$$

where

$$\begin{aligned} \Psi^-(x, t, \varepsilon) = & \int_{-\infty}^0 \exp \left\{ \frac{F^-(y, x, t)}{\varepsilon} \right\} dy, \quad F^-(y, x, t) = -\frac{(y-x)^2}{4(1+t)} + \frac{y^2}{4} + \frac{ay^3}{6}, \\ \Psi^+(x, t, \varepsilon) = & 2\sqrt{\varepsilon\pi(1+t)} - \frac{2\varepsilon(1+t)}{x} \exp \left\{ -\frac{\zeta^2}{4(1+t)} \right\} + \sqrt{\varepsilon(1+t)} \int_{-\infty}^{-\frac{\zeta}{2\sqrt{1+t}}} \frac{e^{-z^2}}{z^2} dz, \end{aligned} \quad (1.6)$$

$$\zeta = x/\sqrt{\varepsilon},$$

$$\Psi^0(x, t, \varepsilon) = \int_{-\infty}^0 (y + ay^2) \exp \left\{ \frac{F^-(y, x, t)}{\varepsilon} \right\} dy.$$

The integral $\Psi^-(x, t, \varepsilon)$ can be written in the form of the sum

$$\Psi^-(x, t, \varepsilon) = \Psi_s^-(x, t, \varepsilon) + \Psi_b^-(x, t, \varepsilon) = \int_{-\infty}^{y^+(x, t)} e^{F^-/\varepsilon} dy + \int_{y^+(x, t)}^0 e^{F^-/\varepsilon} dy,$$

where $y^+(x, t)$ is the point of a local minimum of $F^-(y, x, t)$ in the variable y . The asymptotics of $\Psi_s^-(x, t, \varepsilon)$ is found by Laplace's method. Therefore, we need expressions

$$y^\mp(x, t) = -\frac{t \pm \sqrt{t^2 - 4ax(1+t)}}{2a(1+t)} \quad (1.7)$$

for extremal points of the function F^- in the variable y (for fixed x and t) and values of the function F^- and its derivatives at these points.

Substituting $y = y^\mp(x, t)$ into expressions for $F_{yy}^-(y, x, t)$ and $F^-(y, x, t)$, we have

$$F_{yy}^-(y^\mp(x, t), x, t) = \mp \frac{R(x, t)}{2(1+t)}, \quad \text{where} \quad R(x, t) = \sqrt{t^2 - 4ax(1+t)}, \quad (1.8)$$

$$F^-(y^\mp(x, t), x, t) = -\frac{xt + 3ax^2(1+t) - \frac{t \pm R(x, t)}{2a(1+t)}[R(x, t)]^2}{12a(1+t)^2}. \quad (1.9)$$

Thus,

$$\Psi_s^-(x, t, \varepsilon) = \sqrt{2\pi\varepsilon} \exp \left[\frac{F^-(y^-(x, t), x, t)}{\varepsilon} \right] \sum_{j=0}^{\infty} \varepsilon^j \gamma_j a^{2j} [H(x, t)]^{6j+1}, \quad (1.10)$$

where

$$H(x, t) = \frac{1}{\sqrt{-F_{yy}^-(y^-(x, t), x, t)}}, \quad \gamma_0 = 1, \quad \gamma_1 = 5/24.$$

Let us introduce a stretched variable $\sigma = \frac{x - s(t)}{\varepsilon}$, where $s(t)$ is from (1.3). We make the change $x = s(t) + \varepsilon\sigma$ in expressions for $R(x, t)$, $H(x, t)$, and $y^\pm(x, t)$:

$$R(x, t) \Big|_{x=s(t)+\varepsilon\sigma} = \frac{[r(t)]^2}{3} + \sum_{m=1}^{N-1} (\sigma\varepsilon)^m r_m(t) + O(\varepsilon^{\alpha N}), \quad r_m(t) = O(t^{1-2m}), \quad (1.11)$$

$$H(x, t) \Big|_{x=s(t)+\varepsilon\sigma} = \sqrt{\frac{2(1+t)}{R(s(t) + \varepsilon\sigma, t)}} = \sqrt{6(1+t)} \sum_{m=0}^{N-1} c_m \frac{[a(1+t)\sigma\varepsilon]^m}{[r(t)]^{4m+1}} + O(\varepsilon^{\alpha N}), \quad (1.12)$$

where $r(t) = \sqrt{3t + 4 - 2\sqrt{3t + 4}}$, $c_0 = 1$, $c_1 = 9$;

$$y^\pm(x, t) \Big|_{x=s(t)+\varepsilon\sigma} = \sum_{m=0}^{N-1} (\sigma\varepsilon)^m Y_m^\pm(t) + O(\varepsilon^{\alpha N}), \quad Y_m^\pm(t) = O(t^{1-2m}). \quad (1.13)$$

Using expansion (1.11), let us pass to variable σ in formula (1.9)

$$F^-(y^-(x, t), x, t) \Big|_{x=s(t)+\varepsilon\sigma} = -\mu(t)\sigma\varepsilon + \sum_{m=2}^{N-1} (\varepsilon\sigma)^m F_{-,m}^-(t) + O(t^3\varepsilon^{\alpha N}), \quad (1.14)$$

$$F^-(y^+(x, t), x, t) \Big|_{x=s(t)+\varepsilon\sigma} = -\frac{1}{324a^2} \frac{[r(t)]^6}{(1+t)^3} + \sum_{m=1}^{N-1} (\varepsilon\sigma)^m F_{+,m}^-(t) + O(t^3\varepsilon^{\alpha N}), \quad (1.15)$$

$$\text{where } F_{\pm,m}^-(t) = O(t^{3-2m}), \quad \mu(t) = s'(t) = \frac{(3t+1)\sqrt{3t+4}-2}{18a(1+t)^2}.$$

Using expansions (1.10) and (1.12), we obtain

$$\Psi_s^-(x, t, \varepsilon) = \sqrt{2\pi\varepsilon} \exp \left[\frac{F^-(y^-(x, t), x, t)}{\varepsilon} \right] \frac{\sqrt{6(1+t)}}{r(t)} \left\{ 1 + \sum_{m=1}^{N-1} \varepsilon^m \Psi_{s,m}^-(\sigma, t) + O(\varepsilon^{\alpha N}) \right\},$$

where $\Psi_{s,m}^- \in \Pi_m$. To describe the behavior of coefficients of some expansion, we used the class of functions

$$\Pi_m = \{ \Psi \in C^\infty : |\Psi(\sigma, t)| \leq P_m(\sigma t^{-2}, t^{-3}) \text{ } (\sigma, t) \in \mathbb{R}^1 \times (0, T) \}$$

(P_m is a homogeneous polynomial of degree m).

Now, let us consider $\Psi_b^-(x, t, \varepsilon)$. For $t > \varepsilon^{1/4-\lambda}$

$$|\Psi_b^-(x, t, \varepsilon)| \leq |y^+(x, t)| \exp [F^-(0, x, t)/\varepsilon] = O(\exp[-\varepsilon^{-\lambda_0}]), \quad \lambda_0 > 0.$$

For $t < \varepsilon^{1/4-\lambda}$ we represent it in the form sum:

$$\Psi_b^-(x, t, \varepsilon) = \int_{y^+(x, t)}^{-gt^\gamma} e^{F^-/\varepsilon} dy + \int_{-gt^\gamma}^0 e^{F^-/\varepsilon} dy,$$

where $g > 0$, $3/2 < \gamma < 3/(1-\alpha) - 2$. The first integral is estimated as follows:

$$\left| \int_{y^+(x, t)}^{-gt^\gamma} e^{F^-/\varepsilon} dy \right| \leq |y^+(x, t)| \exp [F^-(-gt^\gamma, x, t)/\varepsilon] = O\left(e^{-g_1 t^{2+\gamma}/\varepsilon}\right).$$

The inequality $\gamma < 3/(1-\alpha) - 2$ provides an exponential smallness of the remainder in the domain

$$\Omega_\alpha = \{|\sigma| < t^2 \varepsilon^{4\alpha-1}, \varepsilon^{1-\alpha} < t^3 < \text{const}, 0 < \alpha < 1\}.$$

From the inequality $\gamma > 3/2$ we conclude that for $-gt^\gamma \leq y \leq 0$ and $t < \varepsilon^{1/4-\lambda}$

$$\delta(y, t) \equiv \frac{1}{\varepsilon} \left[\frac{ty^2}{4(1+t)} + \frac{ay^3}{6} \right] = O(\varepsilon^{\lambda_1}), \quad \text{where } \lambda_1 > 0.$$

This allows one to expand $e^{F^-/\varepsilon}$ in the second integral into a Taylor series in the small parameter $\delta(y, t)$:

$$\Psi_b^- = \exp \left\{ -\frac{\zeta^2}{4(1+t)} \right\} \left[\sum_{m=1}^{N-1} \varepsilon^m \sum_{l=0}^{[(m-1)/2]} a_{m,l} \frac{t^{m-1-2l}(1+t)^{m+l}}{x^{2m-1-l}} + O(t\varepsilon^{\alpha N}) \right],$$

where, in particular, $a_{1,0} = 2$, $a_{2,0} = 4$. Let us pass to variable σ :

$$\Psi_b^- = \exp \left\{ -\frac{\zeta^2}{4(1+t)} \right\} \left[\frac{2\varepsilon(1+t)}{x} + \sum_{m=2}^{N-1} \varepsilon^m \Psi_{b,m}^-(\sigma, t) + O(\varepsilon^{\alpha N}) \right], \quad \Psi_{b,m}^- \in \Pi_m.$$

Thus, we obtain

$$\begin{aligned} \Psi^-(x, t, \varepsilon) &= \sqrt{2\pi\varepsilon} \exp \left[\frac{F^-(y^-(x, t), x, t)}{\varepsilon} \right] \frac{\sqrt{6(1+t)}}{r(t)} + \\ &+ \frac{2\varepsilon(1+t)}{x} \exp \left\{ -\frac{\zeta^2}{4(1+t)} \right\} + \exp \left\{ -\frac{\zeta^2}{4(1+t)} \right\} O(t\varepsilon^{2\alpha}). \end{aligned} \quad (1.16)$$

The function Ψ^0 can be represented in the form of the sum: $\Psi^0 = \Psi_s^0 + \Psi_b^0$, where

$$\Psi_s^0(x, t, \varepsilon) = \int_{-\infty}^{y^+(x, t)} (y + ay^2) e^{F^-/\varepsilon} dy, \quad \Psi_b^0(x, t, \varepsilon) = \int_{y^+(x, t)}^0 (y + ay^2) e^{F^-/\varepsilon} dy.$$

Functions Ψ_s^0 and Ψ_b^0 are studied in exactly the same way as Ψ_s^- and Ψ_b^- , respectively. Thus,

$$\Psi_b^0(x, t, \varepsilon) = \exp \left\{ -\frac{\zeta^2}{4(1+t)} \right\} \left[\sum_{m=2}^{N-1} \varepsilon^m \Psi_{b,m}^0(\sigma, t) + O(\varepsilon^{\alpha N}) \right], \quad \Psi_{b,m}^0 \in \Pi_m,$$

$$\Psi_s^0(x, t, \varepsilon) = -\sqrt{2\pi\varepsilon} \exp \left[\frac{F^-(y^-(x, t), x, t)}{\varepsilon} \right] \sum_{j=0}^{\infty} \varepsilon^j \Psi_j^0(x, t),$$

where

$$\Psi_0^0(x, t) = \frac{t + R(x, t) + 2ax(1+t)}{2a(1+t)^2} H(x, t),$$

$$\Psi_j^0(x, t) = q[H(x, t)]^{6j-3} + q \frac{R(x, t) - 1}{(1+t)} [H(x, t)]^{6j-1} + q \frac{t + R(x, t) + 2ax(1+t)}{(1+t)^2} [H(x, t)]^{6j+1}.$$

Passing to variable σ , we obtain

$$\Psi_s^0(x, t, \varepsilon) = -\sqrt{2\pi\varepsilon} \exp \left[\frac{F^-(y^-(x, t), x, t)}{\varepsilon} \right] \frac{\sqrt{6(1+t)}}{r(t)} \left\{ 2\mu(t) + \sum_{m=1}^{N-1} \varepsilon^m \Psi_{s,m}^0(\sigma, t) + O(\varepsilon^{\alpha N}) \right\}, \quad (1.17)$$

where $\Psi_{s,m}^0 \in \Pi_m$.

Using expression (1.6), (1.16), and (1.17), we arrive at the following result [4].

Theorem 1. *In the domain*

$$\Omega_\alpha = \{|\sigma| < t^2 \varepsilon^{4\alpha-1}, \quad \varepsilon^{1-\alpha} < t^3 < \text{const}, \quad 0 < \alpha < 1\}$$

for the solution of problem (1.1)–(1.2), there holds the asymptotic formula

$$u(x, t, \varepsilon) = \sum_{p=0}^{N-1} \varepsilon^{p/2} h_p(\sigma, \zeta, t) + O(\varepsilon^{\bar{N}}), \quad (1.18)$$

where $\bar{N} \rightarrow \infty$ as $N \rightarrow \infty$. The leading terms of this expansion is

$$h_0(\sigma, \zeta, t) = \frac{2\mu(t)}{1 + K(t) \exp(\mu(t)\sigma) \left[1 + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{-\zeta/(2\sqrt{1+t})} \frac{e^{-z^2}}{z^2} dz \right]},$$

$$\text{where } K(t) = \frac{r(t)}{\sqrt{3}} = \sqrt{\frac{t}{2}} \left(1 + \frac{3t}{32} + O(t^2) \right).$$

Expansion (1.18) shows the presence of an additional scale in the form of the stretched variable $\zeta = x/\sqrt{\varepsilon}$.

2 Large initial gradient

In this section, we consider the Cauchy problem for a more general quasilinear parabolic equation [5, 9, 10, 11]:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad \varepsilon > 0, \quad (2.1)$$

$$u(x, 0, \varepsilon, \rho) = \nu(x\rho^{-1}), \quad x \in \mathbb{R}, \quad \rho > 0. \quad (2.2)$$

We assume that the function φ is infinitely differentiable and its second derivative is strictly positive. The initial function ν is bounded and smooth. The interest to the problem under consideration is explained by applications to studying processes of shock waves formation [3, 12, 13].

The asymptotics of the solution as $\varepsilon \rightarrow 0$ and $\mu = \rho/\varepsilon \rightarrow 0$ in the leading approximation has the form

$$u(x, t, \varepsilon, \rho) = h_0 \left(\frac{x}{\rho}, \frac{\varepsilon t}{\rho^2} \right) - R_{0,0,0} \left(\frac{x}{2\sqrt{\varepsilon t}} \right) + \Gamma \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) + O(\mu^{1/2} \ln \mu),$$

where h_0 , $R_{0,0,0}$ and Γ are known functions. Here, a multiscale character of asymptotics arises from the beginning because of another small parameter ρ .

The behavior of the solution of problem (2.1)–(2.2) is mainly determined by the solution of the limit problem

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \quad u(x, 0) = \begin{cases} \nu_0^-, & x < 0, \\ \nu_0^+, & x \geq 0. \end{cases} \quad (2.3)$$

For $\nu_0^- > \nu_0^+$ using the method of characteristics, we find its generalized solution

$$u_{0,0}(x, t) = \begin{cases} \nu_0^-, & x < ct, \\ \nu_0^+, & x > ct, \end{cases} \quad c = \frac{\varphi(\nu_0^+) - \varphi(\nu_0^-)}{\nu_0^+ - \nu_0^-}.$$

This solution is discontinuous on the line of the shock wave $x = ct$.

In paper [11], for problem (2.1)–(2.2) outer expansions

$$U^+(x, t, \varepsilon, \rho) = \nu_0^+ + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \rho^{m-n} \varepsilon^n u_{m,n}^+(x, t), \quad (2.4)$$

$$U^-(x, t, \varepsilon, \rho) = \nu_0^- + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \rho^{m-n} \varepsilon^n u_{m,n}^-(x, t) \quad (2.5)$$

are constructed in domains

$$\Omega_0^+ = \{(x, t) : x > ct + \varepsilon^{1-\delta_0}, \ 0 < \delta_0 < 1\}$$

and

$$\Omega_0^- = \{(x, t) : x < ct - \varepsilon^{1-\delta_0}\},$$

respectively, where

$$u_{m,n}^{\pm}(x, t) = \sum_{s=n}^{m-1} \frac{\alpha_{m,n,s}^{\pm} t^s}{[x - \varphi'(\nu_0^{\pm})t]^{m+s}},$$

$\alpha_{m,n,s}^{\pm}$ are constants.

Now let us construct an asymptotic solution in a neighborhood of the line of discontinuity. First, rewriting outer expansions (2.4) and (2.5) in terms of the inner variable

$$\sigma = \frac{x - ct}{\varepsilon},$$

we obtain

$$U^{\pm}(ct + \varepsilon\sigma, t, \varepsilon, \rho) = \nu_0^{\pm} + \sum_{n=1}^{\infty} \sum_{m=1}^{[n/2]} \mu^m \varepsilon^{n-m} t^{m-n} P_{n-2m}^{\pm}(\sigma), \quad (2.6)$$

where $P_{n-2m}^{\pm}(\sigma)$ are polynomials of degree $n - 2m$. Taking into account the structure of series (2.6) and the structure of the inner expansion

$$H = \sum_{n=0}^{\infty} \mu^n h_n(x/\rho, \omega), \quad \omega = t\varepsilon/\rho^2,$$

$$h_n = \omega^{n/2} \sum_{m=0}^{\infty} \omega^{-m/2} \sum_{l=0}^m (\ln \omega)^l R_{n,m,l} \left(\frac{x}{2\rho\sqrt{\omega}} \right), \quad |\sigma| + \omega \rightarrow \infty,$$

we will construct an asymptotic solution of equation (2.1) in the domain

$$\Omega_3 = \{(x, t) : |x - ct| < \varepsilon^{1-\delta_3}, \quad t > \varepsilon^{1-\gamma_3}, \quad 0 < \gamma_3, \quad \delta_0 < \delta_3 < 1\}$$

in the form of the series

$$V(\sigma, t, \mu, \varepsilon) = v_0(\sigma) + \sum_{n=1}^{\infty} \sum_{m=1}^n \mu^m \varepsilon^{n-m} \sum_{0 \leq p+q \leq n} (\ln \mu)^p (\ln \varepsilon)^q v_{m,n-m,p,q}(\sigma, t),$$

where

$$v_0(\sigma) = \Lambda(\sigma + \varkappa) \tag{2.7}$$

is the leading term, the function Λ is defined by the formula

$$\int_{(\nu_0^+ + \nu_0^-)/2}^{\Lambda(\sigma)} \frac{dv}{\varphi(v) - cv - b} = \sigma,$$

$$c = \frac{\varphi(\nu_0^+) - \varphi(\nu_0^-)}{\nu_0^+ - \nu_0^-}, \quad b = \frac{\nu_0^+ \varphi(\nu_0^-) - \nu_0^- \varphi(\nu_0^+)}{\nu_0^+ - \nu_0^-},$$

\varkappa is constant, which should be determined by the matching procedure.

In a standard way we arrive at the system of equations

$$\frac{\partial^2 v_0}{\partial \sigma^2} + c \frac{\partial v_0}{\partial \sigma} - \frac{\partial \varphi(v_0)}{\partial \sigma} = 0, \tag{2.8}$$

$$L_3 v_{1,0,p,q} = 0, \tag{2.9}$$

$$L_3 v_{m,n-m,p,q} = \frac{\partial v_{m,n-1-m,p,q}}{\partial t} + \frac{\partial Q_{m,n,p,q}}{\partial \sigma}, \tag{2.10}$$

where

$$L_3 v \equiv \frac{\partial^2 v}{\partial \sigma^2} + c \frac{\partial v}{\partial \sigma} - \frac{\partial \varphi'(v_0)v}{\partial \sigma}, \tag{2.11}$$

$$Q_{m,n,p,q} = \sum_{r=2}^n \frac{\varphi^{(r)}(v_0)}{r!} \sum_{\mathfrak{S}} \prod_{k=1}^r v_{m_k, n_k - m_k, p_k, q_k}, \tag{2.12}$$

$$\mathfrak{S} = \{(m_k, n_k, p_k, q_k) : \sum_{k=1}^r m_k = m, \sum_{k=1}^r n_k = n, \sum_{k=1}^r p_k = p, \sum_{k=1}^r q_k = q\}.$$

We should find solutions of the obtained system satisfying the conditions

$$v_{m,s,p,q}(\sigma, t) = V_{m,s,p,q}^{\pm}(\sigma, t) + O(\sigma^{-\infty}), \quad \sigma \rightarrow \pm\infty, \tag{2.13}$$

where

$$V_{m,s,p,q}^{\pm}(\sigma, t) = \begin{cases} t^{-s} P_{s-m}^{\pm}(\sigma), & s \geq m, p = q = 0, \\ 0, & \text{otherwise,} \end{cases} \quad (2.14)$$

the symbol $O(\sigma^{-\infty})$ denotes a function of order $O(\sigma^{-N})$ for any $N > 0$.

Denote by \mathfrak{C} the class of C^∞ -smooth functions $v(\sigma, t)$ for $\sigma \in \mathbb{R}$, $T_1 \leq t \leq T_2$. By \mathfrak{M}^+ we denote the set of functions from \mathfrak{C} such that there hold the inequalities

$$\left| \frac{\partial^{i+j} v(\sigma, t)}{\partial \sigma^i \partial t^j} \right| \leq M_{i,j} \exp(-\gamma \sigma), \quad \forall i, j.$$

By \mathfrak{M}^- we denote an analogous set of function, for which there hold the inequalities

$$\left| \frac{\partial^{i+j} v(\sigma, t)}{\partial \sigma^i \partial t^j} \right| \leq M_{i,j} \exp(\gamma \sigma), \quad \forall i, j.$$

Further, we need the following statement.

Lemma 1. *Let*

$$\begin{aligned} P^- \in \mathfrak{C}, \quad P^+ \in \mathfrak{C}, \quad F \in \mathfrak{C}, \\ F - L_3 P^- \in \mathfrak{M}^-, \quad F - L_3 P^+ \in \mathfrak{M}^+. \end{aligned}$$

Then for the existence of a solution of the problem

$$L_3 v = F, \quad v - P^\pm \in \mathfrak{M}^\pm$$

it is necessary and sufficient the fulfillment of the condition

$$\begin{aligned} & \left\{ \frac{\partial}{\partial \sigma} (P^+ - P^-) + [c - \varphi(v_0)](P^+ - P^-) \right\}_{\sigma=0} = \\ & = \int_{-\infty}^0 [F(\sigma, t) - L_3 P^-(\sigma, t)] d\sigma + \int_0^\infty [F(\sigma, t) - L_3 P^+(\sigma, t)] d\sigma. \end{aligned} \quad (2.15)$$

The proof of this lemma can be found in [3, chapter VI].

Theorem 2. *For $\sigma \in \mathbb{R}$ and $t > 0$ there exist solutions of equations (2.8)–(2.10) such that $v_0 - \nu_0^\pm \in \mathfrak{M}^\pm$ and $v_{m,n-m,p,q} - V_{m,n-m,p,q}^\pm \in \mathfrak{M}^\pm$, where $V_{m,n-m,p,q}^\pm$ are functions (2.14). Under the condition that all $v_{m',n'-m',p',q'}$ for $n' < n$ are already determined, each function $v_{m,n-m,p,q}$ is determined uniquely up to a term $\varkappa_{m,n-m,p,q} v_0'(\sigma)$, where $\varkappa_{m,n-m,p,q}$ is an arbitrary constant.*

Proof. Formula (2.7) gives the function $v_0(\sigma)$ satisfying equation (2.8) and condition $v_0 - \nu_0^\pm \in \mathfrak{M}^\pm$. Thus, problem (2.9), (2.13) has the solution $v_{1,0,p,q}(\sigma, t) = \varkappa_{1,0,p,q}(t) v_0'(\sigma)$. Proceeding by induction, suppose that solutions of problems (2.10), (2.13) for $n < k$ are constructed so that for $n = k$ the problem has a solution

$$v_{m,k-m,p,q}^*(\sigma, t) + \varkappa_{m,k-m,p,q}(t) v_0'(\sigma).$$

Let us find a function $\varkappa_{m,k-m,p,q}(t)$ for which problem (2.10), (2.13) has a solution for $n = k + 1$. Apply Lemma 1, whose conditions are fulfilled for

$$F(\sigma, t) = \frac{\partial v_{m,k-m,p,q}}{\partial t} + \frac{\partial Q_{m,k+1-m,p,q}}{\partial \sigma}, \quad P^\pm(\sigma, t) = V_{m,k+1-m,p,q}^\pm(\sigma, t),$$

to the function $v_{m,k+1-m,p,q}(\sigma, t)$. According to Lemma 1, for the solvability of problems (2.10), (2.13) for $n = k + 1$ it is necessary and sufficient the fulfillment of conditions (2.15). Substituting functions $v_{m,n-m,p,q}(\sigma, t)$ for $n < k$ and

$$v_{m,k-m,p,q}(\sigma, t) = v_{m,k-m,p,q}^*(\sigma, t) + \varkappa_{m,k-m,p,q}(t)v_0'(\sigma)$$

into this equality, we see that the function $\varkappa_{m,k-m,p,q}(t)$ enters in the form

$$\int_{-\infty}^0 \frac{\partial(\varkappa_{m,k-m,p,q}(t)v_0'(\sigma))}{\partial t} d\sigma + \int_0^{\infty} \frac{\partial(\varkappa_{m,k-m,p,q}(t)v_0'(\sigma))}{\partial t} d\sigma = (\nu_0^+ - \nu_0^-) \frac{d\varkappa_{m,k-m,p,q}(t)}{dt}.$$

Thus, condition (2.15) has the form

$$\varkappa'_{m,k-m,p,q}(t) = f(t),$$

where $f(t)$ is a known function. This implies that a solution $v_{m,k-m,p,q}(\sigma, t)$ is determined up to a term $\varkappa_{m,k-m,p,q}v_0'(\sigma)$ and problem (2.10), (2.13) is solvable for $n = k + 1$. Theorem 2 is proved.

3 Dispersive compression wave

Dispersive compression waves are studied in plasma, fluids, and optics [14, 15, 16, 17]. In particular, studying properties of solutions of the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \varepsilon \frac{\partial^3 u}{\partial x^3} = 0, \quad t \geq 0, \quad \varepsilon > 0, \quad (3.1)$$

is of indisputable interest for describing nonlinear wave phenomena. For steplike initial data, an asymptotic solution of the KdV equation (3.1) can be found by the inverse spectral transform method [18]. Under certain restrictions on the initial function, the asymptotic behavior can be studied by the Whitham method, as in [19, 20]. The long-time asymptotic solution with step-like initial data is also analyzed in [21, 22].

Open questions concerning the behavior of solutions are still the subject of attention for modern researches [23]. Some mathematical results about the solution of the problem in various cases can be found in [24] and [25].

Here, we consider results of paper [26] for the Cauchy problem with the initial condition

$$u(x, 0, \varepsilon, \rho) = \Lambda(x\rho^{-1}), \quad t = 0, \quad x \in \mathbb{R}, \quad \rho > 0, \quad (3.2)$$

introduced in the previous section. We assume the fulfillment of the condition

$$\mu = \frac{\rho}{\sqrt{\varepsilon}} \rightarrow 0.$$

In paper [26], an asymptotic solution to problem (3.1)–(3.2) is constructed using renormalization [27, 28, 29]. Let us pass to the inner variables

$$x = \sqrt{\varepsilon} \eta, \quad t = \sqrt{\varepsilon} \theta,$$

since this allows one to take into account all terms in equation (3.1). As a first approximation, we take the solution of the equation

$$\frac{\partial Z}{\partial \theta} + Z \frac{\partial Z}{\partial \eta} + \frac{\partial^3 Z}{\partial \eta^3} = 0, \quad (3.3)$$

with the initial condition

$$Z(\eta, 0) = \begin{cases} \Lambda_0^-, & \eta < 0, \\ \Lambda_0^+, & \eta > 0, \end{cases} \quad (3.4)$$

where $\Lambda_0^\pm = \lim_{s \rightarrow \pm\infty} \Lambda(s)$. As a model of collisionless shock waves, problem (3.3)–(3.4) was studied by A.V. Gurevich and L.P. Pitaevskii in [30].

Let us construct the expansion of the solution in the following form:

$$u(x, t, \varepsilon, \rho) = Z(\eta, \theta) + \mu W(\eta, \theta, \mu) + O(\mu^\alpha), \quad \alpha > 0, \quad (3.5)$$

where the addend $\mu W(\eta, \theta, \mu)$ must eliminate the singularity of Z at the initial moment of time. Then the function W satisfies the linear equation

$$\frac{\partial W}{\partial \theta} + \frac{\partial(ZW)}{\partial \eta} + \frac{\partial^3 W}{\partial \eta^3} = 0. \quad (3.6)$$

Differentiating equation (3.3) with respect to η , we find that the expression

$$G(\eta, \theta) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \frac{\partial Z(\eta, \theta)}{\partial \eta}$$

satisfies equation (3.6). Moreover, G is the Green function, because

$$\lim_{\theta \rightarrow +0} \int_{-\infty}^{\infty} G(\eta, \theta) f(\eta) d\eta = -\frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z(\eta, 0) f'(\eta) d\eta = f(0)$$

for any smooth function f with compact support, thus $G(\eta, 0) = \delta(\eta)$.

Let us choose the solution W in the form the convolution with the Green function G so that the asymptotic approximation would satisfy the initial condition (3.2). As a result, expansion (3.5) becomes

$$u(x, t, \varepsilon, \rho) = U_0(x, t, \varepsilon, \rho) + O(\mu^\alpha),$$

where

$$U_0(x, t, \varepsilon, \rho) = Z(\eta, \theta) + \frac{\mu}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} \frac{\partial Z(\eta - \mu s, \theta)}{\partial \eta} [\Lambda(s) - Z(s, 0)] ds.$$

Integrating by parts, we obtain the asymptotic approximation

$$u(x, t, \varepsilon, \rho) \approx U_0(x, t, \varepsilon, \rho) = \frac{1}{\Lambda_0^+ - \Lambda_0^-} \int_{-\infty}^{\infty} Z\left(\frac{x - \rho s}{\sqrt{\varepsilon}}, \frac{t}{\sqrt{\varepsilon}}\right) \Lambda'(s) ds.$$

Constructing complete asymptotic expansions of the solution near the singular point by the standard matching method may be connected with serious difficulties. In fact, it is

necessary to solve the scattering problem for a recurrence system of partial differential equations with variable coefficients [3]. In addition, the investigation of the shock wave generated by gradient catastrophe shows that the asymptotics of the solution in a neighborhood of a singular point may have a multiscale structure [4] as we have seen in Section 1.

The renormalization approach allows one to construct a uniformly suitable asymptotics in the whole domain of independent variables avoiding difficulties arising from the matching procedure.

In particular, for $\Lambda_0^+ = 0$ and $\Lambda_0^- = a > 0$ the following formula was obtained in paper [26]:

$$u(x, t, \varepsilon, \rho) \approx 2\Lambda\left(\frac{x+at}{\rho}\right) - \Lambda\left(\frac{x-2at/3}{\rho}\right) - \frac{at}{\rho} \int_{-1}^{2/3} \Lambda'\left(\frac{x-aty}{\rho}\right) \left[2 \operatorname{dn}^2\left(\frac{a^{3/2}t\omega(y)}{\sqrt{\varepsilon}}, \sigma(y)\right) + \sigma^2(y) \right] dy,$$

where $\operatorname{dn}(v, \sigma)$ is the elliptic Jacobi function

$$\operatorname{dn}(u, m) = \sqrt{1 - m \sin^2 \varphi}, \quad u = \int_0^{\varphi(u)} \frac{dv}{\sqrt{1 - m \sin^2 v}},$$

$$\omega(y) = \frac{1}{\sqrt{6}} \left\{ y - \frac{1}{3} [1 + \sigma^2(y)] \right\}, \quad 1 + \sigma^2 - \frac{2\sigma^2(1 - \sigma^2)K(\sigma)}{E(\sigma) - (1 - \sigma^2)K(\sigma)} = 3y,$$

$K(\sigma)$ and $E(\sigma)$ are complete elliptic integrals of first and second kind:

$$K(\sigma) = \int_0^{\pi/2} \frac{dv}{\sqrt{1 - \sigma^2 \sin^2 v}} \quad E(\sigma) = \int_0^{\pi/2} \sqrt{1 - \sigma^2 \sin^2 v} dv.$$

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